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# CONVEX POLYTOPES WITHOUT TRIANGULAR FACES

BY

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ABSTRACT

Let $P$ be a convex $d$-polytope without triangular 2-faces. For $j = 0, \ldots, d-1$ denote by $f_j(P)$ the number of $j$-dimensional faces of $P$. We prove the lower bound $f_j(P) \geq f_j(C_d)$ where $C_d$ is the $d$-cube, which has been conjectured by Y. Kupitz in 1980. We also show that for any $j$ equality is only attained for cubes. This result is a consequence of the far-reaching observation that such polytopes have pairs of disjoint facets. As a further application we show that there exists only one combinatorial type of such polytopes with exactly $2d + 1$ facets.

## 1. Introduction

In $d$-dimensional Euclidean space $E^d$, $d \geq 3$, let $\mathcal{P}^d$ be a class of convex $d$-dimensional polytopes. For $P \in \mathcal{P}^d$ and $j = 0, \ldots, d-1$ denote by $f_j(P)$ the number of $j$-dimensional faces of $P$. The question is to determine $\inf\{f_j(P) \mid P \in \mathcal{P}^d\}$ and to characterize those $P \in \mathcal{P}^d$ which have the smallest possible number of $j$-faces for given $j$. Within the class of all convex polytopes, e.g., exactly the simplices have minimal number of vertices or facets.

If $\mathcal{P}^d$ is the class of $d$-polytopes with a given number of facets, this question has been studied for about twenty years (see [9, p. 183]), but little seems to be known (see [7, p. 129]). If in addition the polytopes of $\mathcal{P}^d$ are simple, we have the famous *lower bound theorem* which has been proved in [1], [2] and [3].

An interesting class of polytopes is the class $\mathcal{P}^d_\square$ of $d$-polytopes without triangular 2-faces or, equivalently, without a face which is a pyramid. In 1980 Y. Kupitz conjectured that $\inf\{f_j(P) \mid P \in \mathcal{P}^d_\square\}$ equals the number of $j$-faces of the $d$-cube (see [11]). This has been proved in [4] and [5], but only for simple polytopes resp. for certain values of $j$ such as most results in this field.

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Here the conjecture of Y. Kupitz is proved without any additional assumption. We show

THEOREM 1. *Let P be a d-polytope without triangular faces and let $C_d$ be the d-cube. Then*

$$(*) \qquad f_j(P) \geq f_j(C_d) = 2^{d-j}\binom{d}{j}, \qquad 0 \leq j \leq d-1.$$

*Furthermore, if equality is attained for some $j \in \{0, \ldots, d-1\}$, then P is combinatorially equivalent to $C_d$.*

This yields a simple characterization of the $d$-cube within a general class of polytopes. Note the interesting fact that the minimal polytopes of $\mathcal{P}^d_\square$ are simple.

The abundance of faces of the polytopes of $\mathcal{P}^d_\square$ comes from an interesting property of them, namely, that they have a pair of disjoint facets. We even have

THEOREM 2. *Let P be a d-polytope without triangular faces. Let F be any facet of P. Then there exists a facet F′ disjoint from F.*

By Theorem 1 a polytope of $\mathcal{P}^d_\square$ has at last $2d$ facets. The $(d-2)$-fold $d$-prism (see [9, p. 56]) with basis a pentagon is a polytope of $\mathcal{P}^d_\square$ with $2d+1$ facets. These are the only such polytopes:

THEOREM 3. *Let P be a d-polytope without triangular faces and with exactly $2d+1$ facets. Then P is a $(d-2)$-fold d-prism with basis a pentagon.*

We shall use the following terminology and notation. Let $P$ be any convex $d$-polytope. We put $f_{-1}(P) := 0$ and $f_d(P) := 1$. $P$ is said to be *cubic* if it is combinatorially equivalent to the cube. If all facets of $P$ are cubic, $P$ is called *cubical.* A vertex of $P$ is called *simple* provided it is contained in exactly $d$ facets. A *subfacet* of $P$ is a $(d-2)$-face of $P$.

## 2. Proof of Theorem 2

Let $P$ be a convex polytope and let $\partial P$ be the boundary complex of $P$. A *shelling* of $\partial P$ is a labelling $\{F_1, \ldots, F_m\}$ of the $m = f_{d-1}(P)$ facets of $P$ so that for $s = 2, \ldots, m-1$ the set $F_s \cap \bigcup_{i=1}^{s-1} F_i$ is homeomorphic to a $(d-2)$-ball (see e.g. [13, p. 173]). It is well known that $\partial P$ is shellable. There even exist special shellings:

LEMMA 1. *Let p be a vertex of P. Then there exists a shelling $\{F_1, F_2, \ldots, F_k, \ldots, F_m\}$ of $\partial P$ so that $p \in F_i$ if and only if $1 \leq i \leq k$.*

This is proved, e.g., in [12]. It may also be proved by a small modification of the original proof of shellability in [8], see [6, Lemma 7].

We prove Theorem 2 by duality. Thus we show: Let $P^*$ be a $d$-polytope in which each ($d$-3)-face is contained in at least 4 facets. Let $p$ be any vertex of $P^*$. Then there exists a vertex $p'$ of $P^*$ such that no facet of $P^*$ contains both $p$ and $p'$.

We assume that this is not true. This means that every vertex of $P^*$ is contained in a facet through $p$.

By Lemma 1 there exists a shelling $\{F_1, F_2, \ldots, F_k, \ldots, F_m\}$ of $\partial P^*$ so that $p \in F_i$ if and only if $1 \le i \le k$. We have $k \le m - 2$, since in case $k = m - 1$ the polytope $P^*$ were dual to a pyramid always having a triangular face, in contradiction to the required property of $P^*$. Hence $D := F_{k+1} \cap \bigcup_{i=1}^{k} F_i$ is homeomorphic to a $(d-2)$-ball.

Therefore $D$ contains a $(d-2)$-face $D_1$. This is a proper face of $F_{k+1}$, so $F_{k+1}$ contains at least one vertex $q \notin D_1$. By assumption every vertex of $P^*$ is contained in $\cup_{i=1}^{k} F_i$, thus $q \in D$. Since $D$ is homeomorphic to a $(d-2)$-ball, it follows that $D$ contains a $(d-2)$-face $D_2$ such that $D_1 \cap D_2$ is a $(d-3)$-face $K$.

We have $D_1 = F_{k+1} \cap F_m$, $D_2 = F_{k+1} \cap F_n$ for certain $m, n \le k$ and $m \ne n$. $F_m \cap F_n$ contains $K$ and $p$, and using $p \notin F_{k+1}$ we have $p \notin K$, hence $F_m \cap F_n$ is $(d-2)$-dimensional. So $K$ is contained in the facets $F_m$, $F_n$, $F_{k+1}$ which pairwise intersect in a $(d-2)$-face containing $K$. Consequently the $(d-3)$-face $K$ is contained in precisely these three facets in contradiction to the required property of $P^*$.

## 3. Proof of Theorem 1

We begin with two remarks on $d$-dimensional cubic polytopes $C_d$.

(i) *The numbers $f_j(C_d)$ may be calculated from $f_j(C_2)$ by induction on $d$ according to*

$$f_j(C_d) = 2f_j(C_{d-1}) + f_{j-1}(C_{d-1}), \qquad 0 \le j \le d-1, \quad d \ge 3.$$

(ii) *Let $P$ be a cubical $d$-polytope. If $P$ has two disjoint facets containing all the vertices of $P$, then $P$ is cubic.*

We prove Theorem 1 by induction on the dimension $d$. The corresponding statement for $d = 2$ is obviously true and we assume that Theorem 1 is true in dimension $d - 1 \ge 2$.

For $d \ge 3$ let $P$ be a $d$-polytope without triangular faces. Let $F$ be a facet of $P$, then $F$ also has no triangular faces. So by induction hypothesis we have

$$f_j(F) \ge f_j(C_{d-1}), \qquad 0 \le j \le d-2,$$

and if equality is attained for some $j \in \{0, \ldots, d-2\}$, then $F$ is cubic.

By Theorem 2 for every facet $F_1$ of $P$ there exists a disjoint facet $F_2$ of $P$. A $j$-face of $F_1$ and a $j$-face of $F_2$ are then disjoint $(0 \leq j \leq d-1)$. Furthermore every $(j-1)$-face of $F_1$ $(j \neq 0)$ is contained in a $j$-face of $P$ different from a $j$-face of $F_1$ or $F_2$. From all this and (i) we obtain

$$({}^{*}_{*}) \quad f_j(P) \geq f_j(F_1) + f_j(F_2) + f_{j-1}(F_1) \geq 2f_j(C_{d-1}) + f_{j-1}(C_{d-1}) = f_j(C_d),$$

$$0 \leq j \leq d-1, \qquad d \geq 3,$$

which proves the inequality $(*)$.

Now we assume that equality is attained in $(*)$ for some given $j \in \{0, \ldots, d-1\}$. Then we have equality in $({}^{*}_{*})$ in any position. An equality sign in the second position of $({}^{*}_{*})$ yields $f_j(F_1) = f_j(C_{d-1})$ and $f_{j-1}(F_1) = f_{j-1}(C_{d-1})$. Because $j \in \{0, \ldots, d-2\}$ or $j-1 \in \{0, \ldots, d-2\}$ we have by induction hypothesis that $F_1$ is cubic. Since $F_1$ is any facet of $P$ we see that $P$ is cubical.

An equality sign in the first position of $({}^{*}_{*})$ yields, if $j = 0$, that all the vertices of $P$ are vertices of $F_1$ or $F_2$. Then by (ii) $P$ is cubic.

An equality sign in the first position of $({}^{*}_{*})$ yields, if $j > 0$, that a $j$-face of $P$ is either contained in $F_1$ or in $F_2$ or intersects $F_1$ in a $(j-1)$-face. In this case it also intersects $F_2$ in a $(j-1)$-face, because the roles of $F_1$ and $F_2$ may be changed without loss of generality. So all the vertices of $P$ are vertices of $F_1$ or $F_2$, hence $P$ is cubic by (ii), which completes the proof.

## 4. Proof of Theorem 3

LEMMA 2. *There does not exist a cubical d-polytope with exactly* $2d+1$ *facets.*

PROOF. Let us assume that $P$ were a $d$-polytope with exactly $2d+1$ cubic facets. If $P$ were simple, then $P$ itself were cubic and so would have $2d$ facets. Hence there exists a vertex of $P$ which is not simple; let $F$ be a facet through this vertex. Because $F$ is cubic, $F$ therefore intersects at least $2(d-1)+1$ facets. Since all the facets of $P$ are cubic, $P$ has no triangular faces and so by Theorem 2 there exists a facet $F'$ disjoint from $F$.

Thus the facets of $P$ are the disjoint facets $F$ and $F'$, $2(d-1)$ facets intersecting $F$ in a $(d-2)$-face, and exactly one facet $F_0$ intersecting $F$ in a $<(d-2)$-face. For $d = 3$ the proof is then easily completed. For $d \geq 4$ we observe that each $(d-2)$-face $S$ of $P$ is contained in at least one facet different from $F_0$, thus $S$ is either contained in $F$ or it is disjoint from $F$ or it intersects $F$ in a $(d-3)$-face. But

$F_0$ contains a $(d-2)$-face intersecting $F$ in a $<(d-3)$-face, which leads to a contradiction.

A $(d-2)$-fold $d$-prism with basis a pentagon will be called briefly a *5-prismatic d-polytope.* Thus a 5-prismatic $d$-polytope is a pentagon for $d = 2$, and it has 5 cubic facets and $2d-4$ 5-prismatic facets for $d \geq 3$.

We prove Theorem 3 by induction on the dimension $d$. The corresponding statement for $d = 2$ is obviously true, and we assume that Theorem 3 is true in dimension $d-1 \geq 2$.

So let $P$ be a $d$-polytope without triangular faces and with exactly $2d+1$ facets, $d \geq 3$. By Theorem 2 every facet of $P$ has then at most $2d-1$ subfacets, so every facet is either cubic by Theorem 1 or 5-prismatic by induction hypothesis. Thus a facet of $P$ is 5-prismatic if and only if it has a 5-prismatic subfacet. Furthermore, by Lemma 2, $P$ has at least one 5-prismatic facet $F$.

Then the facets of $P$ are $F$, a facet $F'$ disjoint from $F$ according to Theorem 2, and the facets $F_1, \ldots, F_{2d-1}$ intersecting $F$ in a subfacet. Thus every vertex of $F$ is simple. Then for $d = 3$, $P$ can easily be seen to be 5-prismatic.

So assume $d \geq 4$. Then $F$ has exactly $k := 2(d-1) - 4$ 5-prismatic subfacets, thus without loss of generality $F_1, \ldots, F_k$ are 5-prismatic, and $F_{k+1}, \ldots, F_{k+5} = F_{2d-1}$ may be cubic. $F'$ has at least $2(d-1) \geq 6$ subfacets and so intersects at least one of the $F_1, \ldots, F_k$ in a subfacet, which is 5-prismatic because $F \cap F' = \emptyset$. Hence $F'$ is 5-prismatic.

Thus the vertices of both $F$ and $F'$ are simple. Since any vertex of $F$ or $F'$ is contained in a 5-prismatic subfacet of $F$ resp. $F'$, it is contained in a 5-prismatic facet intersecting both $F$ and $F'$ in a 5-prismatic subfacet. These two facts yield that the graph of $P$ (i.e., the complex of the vertices and edges of $P$) is that of a 5-prismatic polytope. Since by [6] or [10] a simple polytope is uniquely determined by its graph, $P$ is 5-prismatic.

## 5. Concluding remark

From Theorem 2 there immediately follows

THEOREM 4. *Let $P$ be a convex polytope. Then the following properties are equivalent:*

(1) *$P$ has no triangular 2-faces.*

(2) *No face of $P$ is a pyramid.*

(3) *Let $F_j$ be a $j$-face of $P$. Then for every $(j-1)$-face $F_{j-1} \subset F_j$ there exists a disjoint $(j-1)$-face $F'_{j-1} \subset F_j$.*

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